A FUNDAMENTAL SYSTEM OF COVARIANTS OF THE TERNARY CUBIC FORM.

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1. In many different mathematical investigations use is made of covariants of the ternary cubic form F. Less frequent use is made of the further concomitants involving line coördinates, and these will not be discussed here. The complete system of the 34 concomitants was obtained by symbolic methods by Clebsch and Gordan* and simpler by Gundelfinger.† They were exhibited in non-symbolic form by Cayley‡ for the canonical form $\Sigma a_i x_i^3 + 6lx_1x_2x_3$. Certain concomitants are obtained in the texts by Salmon, Elliott, and Weber, but no attempt is made to find a fundamental system.

The object of the present paper is to prove by an elementary method that a fundamental system of covariants of F is given by F, two invariants§ S and T, the Hessian H of F, the bordered Hessian determinant G, and the Jacobian J of F, H, G:

$$6^3H = \begin{vmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{vmatrix}, \qquad 6^2G = \begin{vmatrix} F_{11} & F_{12} & F_{13} & H_1 \\ F_{21} & F_{22} & F_{23} & H_2 \\ F_{31} & F_{32} & F_{33} & H_3 \\ H_1 & H_2 & H_3 & 0 \end{vmatrix}, \tag{1}$$

$$9J = \begin{vmatrix} F_1 & H_1 & G_1 \\ F_2 & H_2 & G_2 \\ F_3 & H_3 & G_3 \end{vmatrix},$$

where F_{ij} denotes $\partial^2 F/\partial x_i \partial x_j$ and H_i denotes $\partial H/\partial x_i$. The method enables us to compute anew the expressions for S and T, and to deduce the syzygy (9) between them and the covariants.

2. The general ternary cubic form is

$$F = a_0x^3 + 3b_0x^2y + 3c_0xy^2 + d_0y^3 + 3(a_1x^2 + 2b_1xy + c_1y^2)z + 3(a_2x + b_2y)z^2 + a_3z^3.$$

The weight of any coefficient is its subscript; the various terms of any seminvariant (§ 3) are of equal weight.

* Math. Annalen, vol. 6, 1873, p. 436.

† *Ibid.*, vol. 4, 1871, p. 144.

‡ Amer. Jour. Math., vol. 4, 1881, p. 4; Coll. Math. Papers, XI, p. 342.

§ Given in full in Salmon's Higher Plane Curves, § 221; Cayley, Coll. Math. Papers, II, p. 325, where, in S, cf^2h is a misprint for cfh^2, while in the 8th line of the 4th column of T, h^2 is a misprint for k^2 in $chijk^2$, and in the 5th line of the 5th column, fil^4 is a misprint for fjl^4. In the third column of the Hessian, cij and fkl are misprints for cfj and gil.

Without altering x or y, replace z by $z + tx + my$. Then F is replaced by a like form with the coefficients

$$a_3' = a_3, \quad a_2' = a_2 + ta_3, \quad b_2' = b_2 + ma_3, \quad a_1' = a_1 + 2ta_2 + t^2a_3,$$
$$b_1' = b_1 + tb_2 + ma_2 + tma_3, \quad c_1' = c_1 + 2mb_2 + m^2a_3, \quad \cdots.$$

Invariants with respect to all such replacements are obtained by eliminating t and m:

$$a_3' = a_3, \quad a_1'a_3' - a_2'^2 = a_1a_3 - a_2^2, \quad b_1'a_3' - a_2'b_2' = b_1a_3 - a_2b_2, \quad \cdots.$$

Apart from a factor which is a power of a_3, these invariants are the values of a_1', b_1', $\cdots$ for $t = -a_2/a_3$, $m = -b_2/a_3$, which give $a_2' = b_2' = 0$.

Hence by the replacement of z by $z - xa_2/a_3 - yb_2/a_3$, F becomes

$$a_3z^3 + 3zQ/a_3 + f/a_3^2, \tag{2}$$

such that the coefficients in

$$Q = Ax^2 + 2Bxy + Cy^2, \qquad f = ax^3 + 3bx^2y + 3cxy^2 + dy^3$$

are invariants of F with respect to all the transformations

$$z' = z + tx + my, \tag{3}$$

and, conversely,* any polynomial invariant under these transformations is the quotient of a polynomial in a_3, A, $\cdots$, d by a power of a_3. We find that

$$\begin{aligned} A &= a_1a_3 - a_2^2, & a &= a_0a_3^2 - 3a_1a_2a_3 + 2a_2^3,\\ B &= b_1a_3 - a_2b_2, & b &= b_0a_3^2 - a_1b_2a_3 - 2b_1a_2a_3 + 2a_2^2b_2,\\ C &= c_1a_3 - b_2^2, & c &= c_0a_3^2 - 2b_1b_2a_3 - c_1a_2a_3 + 2a_2b_2^2,\\ & & d &= d_0a_3^2 - 3c_1b_2a_3 + 2b_2^3. \end{aligned} \tag{4}$$

3. By a seminvariant of F is meant a homogeneous isobaric polynomial in its coefficients which is invariant with respect to all transformations (3) as well as all linear transformations on x and y. Hence the seminvariants are functions of a_3 and the simultaneous invariants of Q and f.

A fundamental system of invariants of Q and f is known† (§ 7) to be formed by the following five invariants: the discriminant $\Delta = AC - B^2$ of Q, the discriminant

$$D = (ad - bc)^2 - 4(ac - b^2)(bd - c^2)$$

of f, the intermediate invariant‡

$$I = A(bd - c^2) - B(ad - bc) + C(ac - b^2)$$

* For binary forms, cf. Dickson, Algebraic Invariants, 1914, p. 47.

† Dickson, *ibid.*, p. 61; Salmon, Modern Higher Algebra, 4th ed., p. 187.

‡ That of Q and $Q' = A'x^2 + 2B'xy + C'y^2$ is $AC' - 2BB' + CA'$, given by the invariance of the discriminant of $Q + kQ'$.

between Q and the Hessian of f, the resultant R of Q and f, and the resultant M of two linear covariants, R and M being given in full by Salmon. They are connected by the syzygy

$$(5) \quad M^2 = -4\Delta^3D^2 + D(R^2 + 12R\Delta I + 24\Delta^2I^2) - 4RI^3 - 36\Delta I^4.$$

4. The expression obtained from $\Delta = AC - B^2$ by inserting the values (4) is seen to be divisible by a_3, the quotient being

$$(6) \qquad h = \begin{vmatrix} a_1 & b_1 & a_2 \\ b_1 & c_1 & b_2 \\ a_2 & b_2 & a_3 \end{vmatrix},$$

which is the leader (coefficient of z^3) of the Hessian of F. Similarly, we seek other combinations of Δ, D, I, R, M which are divisible by powers of a_3, in order to deduce a fundamental system of seminvariants. But to verify a relation between seminvariants, it is sufficient to prove it for the case in which

$$(7) \qquad a_2 = b_2 = a_1 = c_1 = 0,$$

since F can be transformed into a form satisfying (7) by means of transformations which leave all seminvariants unaltered; after obtaining (2), we have only to introduce the factors of Q as new variables x and y. For (7), we have

$$\Delta = -b_1^2a_3^2, \quad D = \{(a_0d_0 - b_0c_0)^2 - 4(a_0c_0 - b_0^2)(b_0d_0 - c_0^2)\}a_3^8,$$
$$I = -b_1(a_0d_0 - b_0c_0)a_3^5, \; R = -8a_0d_0b_1^3a_3^7, \; M = 8(a_0c_0^3 - b_0^3d_0)b_1^3a_3^{11},$$

while S, T and the leaders (§ 5) g, j of covariants G, J become

$$\begin{aligned} S &= a_0d_0a_3b_1 - b_1a_3b_0c_0 - b_1^4, \\ T &= D/a_3^6 - (20a_0d_0 + 12b_0c_0)b_1^3a_3 - 8b_1^6, \\ g &= 8a_3^3b_0b_1^3c_0 + 9a_3^2b_1^6, \\ j &= -8a_3^5b_1^3(a_0c_0^3 - b_0^3d_0). \end{aligned}$$

By (6), $h = -a_3b_1^2$. Hence we have the relations*

$$(8) \quad \begin{aligned} \Delta &= a_3h, \quad a_3^4S = -I - \Delta^2, & a_3^6T &= D + 12\Delta I + 4R + 8\Delta^3, \\ a_3^4g &= -8\Delta I - R - 9\Delta^3, & a_3^6j &= -M. \end{aligned}$$

Since any seminvariant of F is the quotient of a polynomial in a_3, Δ, I, R, D, M by a power of a_3, it equals the quotient of a polynomial in a_3, h, S, g, T, j by a power of a_3. We may assume that the exponent of j is 0 or 1 in view of (5), or the equivalent syzygy obtained by inserting the values of Δ, I, R, D, M, and noting that the terms in a_3^9, a_3^{10}, a_3^{11} cancel:

* These were also verified for the case $a_2 = b_2 = b_0 = c_0 = 0$.

$$
(9)\qquad
\begin{aligned}
j^2 = &- 4a_3^5hS^4 - 4a_3^4(gS^3 + 2h^2S^2T)\\
&+ a_3^3(108h^3S^3 - 4ghST - 4h^3T^2)\\
&+ a_3^2(36gh^2S^2 + 108h^4ST + g^2T)\\
&- a_3(516h^5S^2 + 36g^2hS + 18gh^3T)\\
&+ 108h^4gS - 27h^6T + 4g^3.
\end{aligned}
$$

The syzygy between the covariants is derived by replacing a_3, h, g, j by F, H, G, J.

To conclude that a fundamental system of seminvariants of F is given by a_3, h, g, j, S, T, it now suffices to verify that no polynomial in the last five, linear in j, is divisible by a_3. It suffices to show this when $a_1 = a_3 = b_1 = b_2 = c_0 = 0$, for which (§ 5)

$$
\begin{aligned}
h &= - a_2^2c_1, \quad g = a_2^6d_0^2, & j &= (- 2a_2^2d_0^3 - 27b_0c_1^4)a_2^7,\\
S &= a_0a_2c_1^2 + a_2^2b_0d_0, & T &= 4a_0a_2^3d_0^2 - 27a_2^2b_0^2c_1^2.
\end{aligned}
$$

No polynomial in h, g, S, T is identically zero, since the Jacobian of S and T with respect to a_0 and b_0 is not identically zero. Next, if $j\rho + \sigma \equiv 0$, where ρ and σ are polynomials in h, g, S, T, we find by changing the signs of b_0 and d_0 that $- j\rho + \sigma \equiv 0$, whence $\sigma \equiv \rho \equiv 0$. Since a covariant is uniquely determined by its leader, which is a seminvariant, the covariants mentioned in § 1 form a fundamental system.

5. To compute the leaders g and j of our covariants G and J, we need certain coefficients of the Hessian:

$$H = Ex^2z + Fxyz + Py^2z + Qxz^2 + Kyz^2 + Lz^3 + \cdots.$$

Then the coefficient of z^6 in G is

$$
\begin{gathered}
g = Q^2\gamma + 2QK\delta + K^2\epsilon + 6QL\kappa + 6LK\lambda + 9L^2\mu,\\
\gamma = b_2^2 - a_3c_1, \quad \delta = a_3b_1 - a_2b_2, \quad \epsilon = a_2^2 - a_1a_3, \quad \kappa = a_2c_1 - b_1b_2,\\
\lambda = a_1b_2 - a_2b_1, \quad \mu = b_1^2 - a_1c_1.
\end{gathered}
$$

The coefficients of xz^5 and yz^5 in G are respectively

$$
\begin{aligned}
v = &\; Q^2(2b_1b_2 - a_2c_1 - a_3c_0) + 4EQ\gamma + 2QK(a_3b_0 - a_1b_2) + (2QF + 4EK)\delta\\
&+ K^2(a_1a_2 - a_0a_3) + 2FK\epsilon + 6QL(a_2c_0 - b_0b_2 + \mu) + (4Q^2 + 12EL)\kappa\\
&+ 6LK(a_0b_2 - a_2b_0) + (6LF + 4QK)\lambda + 9L^2(2b_0b_1 - a_0c_1 - a_1c_0),\\
w = &\; Q^2(b_2c_1 - a_3d_0) + 2QF\gamma + 2QK(a_3c_0 - a_2c_1) + (2FK + 4QP)\delta\\
&+ K^2(2a_2b_1 - a_1b_2 - a_3b_0) + 4PK\epsilon + 6QL(a_2d_0 - b_2c_0)\\
&+ (4QK + 6FL)\kappa + 6LK(b_0b_2 - a_2c_0 + \mu)\\
&+ (12LP + 4K^2)\lambda + 9L^2(2b_1c_0 - a_1d_0 - b_0c_1).
\end{aligned}
$$

Then

$$3j = \begin{vmatrix} a_2 & Q & v \\ b_2 & K & w \\ a_3 & 3L & 6g \end{vmatrix}.$$

6. If we seek all the concomitants of F, viz., the covariants of F and a linear form L, let the transformation which reduces F to (2) replace L by $kz + l$, where l is linear in x, y. Hence we need the invariants of l, Q, f, viz., the covariants (or seminvariants) of Q and f. Although the latter are known (§ 7) and various concomitants of F can be readily deduced, the work of deriving a fundamental system and especially the proof that it is complete would seem prohibitive by this method.

7. If we seek a fundamental system of seminvariants of the binary quadratic form Q and cubic form f, given in § 2, we begin by removing the second term of f by replacing x by $x - yb/a$. Then f and Q become

$$ax^3 + \frac{3}{a}A_{22}xy^2 + \frac{1}{a^2}A_{33}y^3, \qquad Ax^2 - \frac{2}{a}B_{13}xy + \frac{1}{a^2}(aB_{11} - AA_{22})y^2,$$

where

$$A_{22} = ac - b^2, \quad A_{33} = a^2d - 3abc + 2b^3, \quad B_{13} = Ab - Ba, \quad B_{11} = Ac - 2Bb + Ca.$$

Hence every seminvariant is the quotient of a polynomial in $A_{13} = a$, A_{22}, A_{33}, $B_{02} = A$, B_{11}, B_{13} by a power of A_{13}. Among these quotients is the discriminant $A_{40} = D$ of f given by the syzygy

$$A_{13}^2A_{40} - 4A_{22}^3 - A_{33}^2 = 0.$$

Other quotients B_{22}, B_{31}, $B_{20} = I$, C_{11}, $C_{00} = \Delta$, $C_{31} = L_4 + IL_1$, $D_{20} = R + 8\Delta I$, $D_{40} = M$ are defined in turn by Hammond's* syzygies (2), (3), (4), (8), (9), (13), (23), (27) between our 15 seminvariants. He listed 35 further syzygies deducible from these nine. These 44 syzygies might be used to simplify any polynomial in the 15 seminvariants in an attempt to prove that the simplified polynomial, regarded as a function of $A, \cdots, d$, does not have the factor $A_{13} = a$, unless the initial polynomial has the explicit factor A_{13}, and hence to prove that the 15 forms give a fundamental system. To indicate only one step in this rather prohibitive work, we first eliminate the products of D_{40} by A_{13}, A_{22}, A_{33}, B_{02}, B_{11}, B_{13}, B_{22}, B_{31}, C_{11}, C_{31}, D_{40} by means of syzygies (27), (30), (32), (35), (37), (38), (39), (41), (42), (43), (44); then D_{40} occurs only with invariants A_{40}, B_{20}, C_{00}, D_{20}. Since D_{40} is the only one of these invariants which is skew (of odd weight), it cannot occur in the polynomial.

Such a proof, if completed, would also yield a complete set of syzygies. The proof that the 15 covariants form a fundamental system is however much simpler by the symbolic theory.†

* Amer. Jour. Math., vol. 8, 1886, p. 138. His notation for a covariant has been retained here for its seminvariant leader. In verifying a syzygy between the latter, we may take $b = 0$.

† Clebsch, Binären Algebraischen Formen, 1872, p. 209; Glenn, The Theory of Invariants, 1915, p. 146.